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DISCRETE APPROXIMATIONS RELATED TO NONLINEAR THEORIES OF SOLIDS

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Abstract—Interpolation and extrapolation are employed to approximate the fields in a nonlinear theory of solid bodies. Nodal points are employed in the space of position and load, and the continuous fields are essentially replaced by nodal values. Interpolation between nodes (extrapolation in load) defines a continuous approximation. The differential equations of the continuum are replaced by algebraic equations of the discrete system. Nonlinear equations are replaced by a succession of linear equations, as a nonlinear path is approximated by linear segments.

Variational theorems are used as the bases of the algebraic formulations which govern the discrete approximation. The algebraic equations are related to their differential counterparts.

A generalized arc-length is introduced in the configuration-load space in order to facilitate the incremental computations near limit points. The arc-length is used as the loading parameter in some illustrative problems.

An appendix describes the viewpoint of finite elements and the continuity conditions which insure the equivalence of the methods.

NOTATION

Index notations and the summation convention are employed. Latin minuscules signify a spatial coordinate θ^i and represent the numbers 1, 2, 3. Greek minuscules signify a surface coordinate θ^α and represent the numbers 1, 2. Latin majuscules signify a nodal quantity and represent the number assigned to the node. Indices enclosed by parentheses are not summed and underlined subscripts are not indices. A comma signifies partial differentiation. In general, the notations follow the texts of Green and Zerna [9] and Green and Adkins [10]. Specific notations follow:

θ^i	coordinate ($i = 1, 2, 3$)
$\mathbf{r}, \mathbf{R}$	position vector of undeformed, deformed body
$\mathbf{g}_i$	$= \mathbf{r}_{,i} \equiv \partial \mathbf{r}/\partial \theta^i$
g_{ij}	$= \mathbf{g}_i \cdot \mathbf{g}_j =$ component of metric tensor of undeformed body
G_{ij}	$= \mathbf{G}_i \cdot \mathbf{G}_j =$ component of metric tensor of deformed body
g	$= \lvert g_{ij} \rvert =$ determinant of g_{ij}
G	$= \lvert G_{ij} \rvert =$ determinant of G_{ij}
$\mathbf{V}$	displacement vector
$\mathbf{t}$	stress vector (per unit undeformed area)
$\mathbf{G}^i$	reciprocal base vector $\mathbf{G}^i \cdot \mathbf{G}_j \equiv \delta^i_j$
$\hat{n}$	unit normal to undeformed surface
${}_0n_i$	$\equiv \mathbf{g}_i \cdot \hat{n}$
${}_0n^i$	$\equiv \mathbf{g}^i \cdot \hat{n}$
s^{ij}	$= \mathbf{t} \cdot \mathbf{G}^j {}_0n^i$
ρ_0	mass density of undeformed body

INTRODUCTION

THE essential difficulty in the analysis of continuous structures rests with our inability to solve the governing differential equations. These difficulties are often insurmountable if the equations are nonlinear. On the other hand, the development of computers has provided a great capability for the numerical solution of large systems of linear algebraic equations.

Accordingly, our attention is directed toward means of utilizing the latter capability for numerical treatment of the nonlinear problems.

The approximation of a continuous function by a spline function [1] is an extension of the notion that any smooth curve can be approximated by *interpolating* curves within small segments. The method of finite elements [2, 3] is a means to construct spline approximations. A continuous function of position is replaced by a finite set of nodal values, and a differential equation is replaced by an algebraic equation.

The variables (continuous functions or finite sets) which define the configuration of the body are expected to change continuously with loading (or time) except at points of instability. The state of the system is defined by the variables of configuration and load; the successive states trace a path (in function space or in n-dimensional space). A nonlinear path can be approximated by a succession of linear segments; each segment corresponds to an increment in a loading parameter [4, 5] and each step requires the solution of linear equations.

METHODS OF EXTRAPOLATION AND INTERPOLATION

Approximating the loading path

The ideas of interpolation and extrapolation provides the means to approximate any continuous function. In most problems of structural mechanics we wish to approximate the displacement vector **V** according to a continuum theory. The displacement depends upon the coordinates of position $\theta^i (i = 1, 2, 3)$ and the loading. For simplicity, we assume a proportional loading wherein the magnitude of the loads is determined by a parameter λ as depicted in Fig. 1. Then

$$\mathbf{V} = \mathbf{V}(\theta^1, \theta^2, \theta^3; \lambda). \tag{1}$$

In the event of large rotations, large strains or a nonlinear material, the displacement of a given particle is a nonlinear function of the load parameter as shown in Fig. 2. The governing equations are nonlinear, for example, the equation governing the finite deformations of an elastic body. However, if a body is in a stable state of equilibrium, then the equations governing a small perturbation are linear, typically

$$F_i(\bar{V}^1, \bar{V}^2, \bar{V}^3; \lambda)\Delta V^i + F_\lambda(\bar{V}^1, \bar{V}^2, \bar{V}^3; \lambda)\Delta\lambda = 0 \tag{2}$$

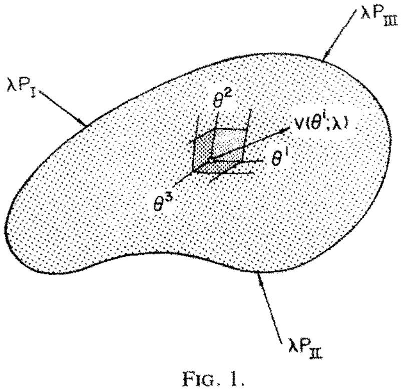

FIG. 1.

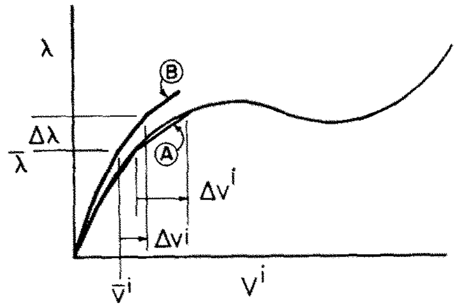

FIG. 2.

where F_i and F_λ are linear operators which depend on the reference state $(\bar{V}^1, \bar{V}^2, \bar{V}^3; \lambda)$ and operate on the increments ΔV^i and $\Delta\lambda$. Strictly speaking, such equations govern only infinitesimal changes of state $(\Delta V^1, \Delta\lambda \to 0)$, but may hold also for inelastic deformations. If (2) has a nontrivial solution $\Delta\tilde{V}^j$, $\Delta\lambda = 0$, then the reference state is "critical", a branch point or limit point in the loading path; $\Delta\tilde{V}^j$ is a "buckling mode".

After continuous loading, the body suffers finite deformations governed by functional equations of the form:

$$F \equiv \int_{(0,0,0;0)}^{(V^1,V^2,V^3;\lambda)} (F_i\,\mathrm{d}V^i + F_\lambda\,\mathrm{d}\lambda) = 0. \tag{3a}$$

If the body is elastic, then the governing equation (3a) depends solely upon the state variables, that is

$$F = N(V^1, V^2, V^3; \lambda) = 0 \tag{3b}$$

where N is a nonlinear operator.

If the operators of (2) are continuous, then mean values should produce finite, but small, increments $(\Delta V^1, \Delta V^2, \Delta V^3; \Delta\lambda)$ which would interpolate the actual path as depicted at A of Fig. 2. Unfortunately, such mean values are unpredictable, but values of the reference state can be employed to extrapolate the path. However, the approximation can stray from the path, as shown at B; such deviation depends upon the relative size of the steps. Corrections are also possible from *a priori* estimates of the mean values [11, 12] or by *a posteriori* correction of the residual error.

In principle, incremental loading is extrapolation and serves to replace the nonlinear differential equations of a continuum theory by a succession of linear equations. However, in practice, the successive differential equations are progressively more complicated and are usually intractable. Consequently, it is also necessary to approximate the function $\mathbf{V}$ with respect to the coordinates θ^i. Such approximation leads to linear algebraic equations governing the incremental displacement.

Approximating functions of position

Any continuous field can be approximated by discrete nodal values with interpolation between nodes. When the interpolating functions are polynomials the approximating function is termed a spline function [1, 13]. In practice, we are interested in such approximations of unknown solutions of the differential or integral equations governing solid bodies.

In the continuum theory we seek those continuous functions which satisfy the governing equations. If the system is conservative, we require the continuous displacement which achieves a minimum value of the energy functional. In our approximation, we seek the finite set of nodal values which determines the best approximation of the continuous function in accordance with preselected interpolating functions. If the system is conservative, then the "best" approximation is that which achieves the minimum from the subclass of spline functions. The procedure is a straightforward, albeit tedious, application of the Rayleigh–Ritz [14, 15] procedure. If the system is non-conservative, the continuous solution can be achieved by averaging in the manner of Galerkin [16]. By one scheme or another, any theory of continua can be converted to a theory of finite sets.

The principle of minimum potential energy has served as the vehicle for most approximations formed with a variety of interpolating polynomials. Then the energy convergence follows and, if the body is Hookean, convergence of the field can be proven [17–20]. If the system is non-conservative, the discrete theory may be achieved by the principle of virtual work or the Galerkin procedure. The theorem of minimum complementary energy has also served as the basis of discrete formulations for the Hookean body and, again, the convergence can be shown [21]. The stationary theorem of Hellinger–Reissner can also serve as the basis of the approximation [22, 23]. However, when the stationary condition does not imply an extremum, then the proof of convergence may not follow.

In the following we review the various energy principles as the bases of spline approximations of the continua. Particular attention is given to the interpretation of the equations governing the discrete approximation.

Commutation of the approximating procedures

Two forms of approximation are needed to replace the *nonlinear differential* equations of a continuum theory by a succession of *linear algebraic* equations, namely, the methods of incremental loading (extrapolation) and finite-elements (interpolation). In principle, the procedures can be applied in either order. If the former is applied first, the result is a succession of linear differential equations; subsequent spline approximations of the continuous fields yield linear algebraic equations. In some instances, this order is the most practical, especially, if the continuum theory has been formulated from the variational viewpoint; for example, the work of Biot [4] or the stability theory of Koiter [24]. However, there appears to be one advantage to approximating the spatial field first and then linearizing via incremental loads: if the nonlinear algebraic equations are available, it is possible to assess accumulative errors and to introduce corrective steps.

APPROXIMATING THE FIELDS

Approximate conditions of equilibrium (or motion)

In continuum theories of solids, a stress field satisfies equilibrium if the virtual work vanishes for an *arbitrary* but *continuous* variation $\delta\mathbf{V}$ of the displacement field:

$$\iiint_v (s^{ij}\mathbf{G}_j \cdot \delta\mathbf{V}_{,i})\,dv - \iiint_v (\rho_0\mathbf{f} \cdot \delta\mathbf{V})\,dv - \iint_s \mathbf{t} \cdot \delta\mathbf{V}\,ds = 0. \tag{4a}$$

Here v and s signify the volume and surface of a reference state, s^{ij} the stress component per unit area of that reference state and ρ_0 the mass density of that state. Apart from the absence of couple-stress, the statement (4a) is general; it applies to finite deformation.

Upon integrating the first integral of (4a) by parts we obtain

$$-\iiint_v \left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta^i}(\sqrt{g}s^{ij}\mathbf{G}_j)+\rho_0\mathbf{f}\right]\cdot\delta\mathbf{V}\,\mathrm{d}v+\iint_s [{}_0n_is^{ij}\mathbf{G}_j-\mathbf{t}]\cdot\delta\mathbf{V}\,\mathrm{d}s = 0. \tag{4b}$$

Following the usual arguments that $\delta\mathbf{V}$ is arbitrary in v and on s, we arrive at the differential equations and boundary conditions: the bracket of the first integral must vanish at *each point* within v and the bracket of the second integral must vanish at *each point* on s. We reiterate these well-known statements of the continuum theory in order that we may later draw the analogy with the statements of the discrete theory.

A first step in the approximation of the spatial fields consists of subdividing the body into finite elements. Often the body is conveniently partitioned into quadrilateral elements by coordinate surfaces as shown in Fig. 3. Then a typical interior node I is a particle at the contiguous corners of eight elements as shown in Fig. 4. The region occupied by these eight

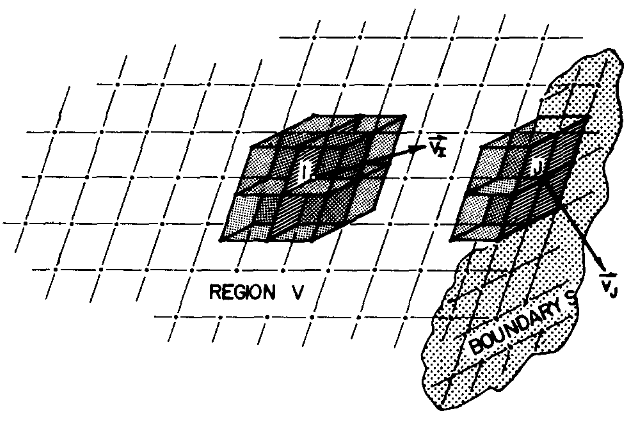

FIG. 3.

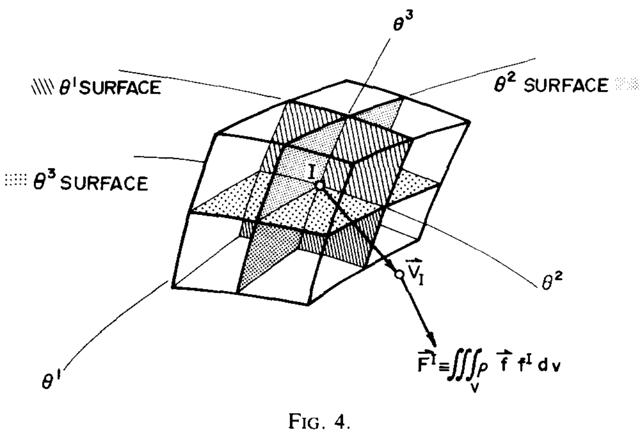

FIG. 4.

elements in the reference state is designated v_{I}. The simplest spline approximation of the displacement has the form:

$$\mathbf{V} = f^{\mathrm{I}}(\theta^1, \theta^2, \theta^3)\mathbf{V}_{\mathrm{I}}. \tag{5}$$

The function f^{I} is continuous in v_{I} and vanishes on the surface of v_{I}. The function f^{I} is based upon an interpolation of $\mathbf{V}$ between nodes: the simplest polynomial approximation in the first quadrant yields

$$f^{\mathrm{I}} = 1 - \frac{\theta^1}{h^1} - \frac{\theta^2}{h^2} - \frac{\theta^3}{h^3} + \frac{\theta^1\theta^2}{h^1h^2} + \frac{\theta^1\theta^3}{h^1h^3} + \frac{\theta^2\theta^3}{h^2h^3} - \frac{\theta^1\theta^2\theta^3}{h^1h^2h^3}, 0 \leq \theta^i \leq h^i. \tag{6}$$

Here the origin is placed at the node and $\theta^i = 0$, h^i define the boundaries of the first quadrant. With similar interpolation throughout v_{I}, we have

$$\int\int\int_{v_{\mathrm{I}}} f^{\mathrm{I}} \,\mathrm{d}\theta^1 \,\mathrm{d}\theta^2 \,\mathrm{d}\theta^3 = 1. \tag{7a}$$

If s_i denotes the interface along the θ^i surface through node I,

$$\int\int_{s_i} f^{\mathrm{I}} \,\mathrm{d}\theta^j \,\mathrm{d}\theta^k = 1 \qquad (i \neq j \neq k \neq i). \tag{7b}$$

According to (5), the virtual displacement $\delta\mathbf{V}$ is no longer arbitrary: now only discrete nodal values $\delta\mathbf{V}_{\mathrm{I}}$ are arbitrary. In place of (4a), we have

$$\delta\mathbf{V}_{\mathrm{I}} \cdot \left\{\int\int\int_{v} (s^{ij}\mathbf{G}_j f^{\mathrm{I}}_{,i}) \,\mathrm{d}v - \int\int\int_{v} (\rho_0 \mathbf{f} f^{\mathrm{I}}) \,\mathrm{d}v - \int\int_{s} \mathbf{t} f^{\mathrm{I}} \,\mathrm{d}s\right\} = 0. \tag{8a}$$

Because f^{I} vanishes outside of v_{I}, each integral extends only through v_{I}, depicted in Fig. 4. Since f^{I} has continuous derivatives within each element, we can integrate-by-parts to obtain

$$\delta\mathbf{V}_{\mathrm{I}} \cdot \left\{-\int\int\int_{v_{\mathrm{I}}} \left[\frac{1}{\sqrt{g}} \frac{\partial}{\partial\theta^i}(s^{ij}\mathbf{G}_j) + \rho_0 \mathbf{f}\right] f^{\mathrm{I}} \,\mathrm{d}v + \int\int_{s_{(i)}} [{}_0n_i(s^{ij}_{+} - s^{ij}_{-})\mathbf{G}_j] f^{\mathrm{I}} \,\mathrm{d}s_{(i)} \right.$$
$$\left. + \int\int_{s} [{}_0n_i s^{ij}\mathbf{G}_j - \mathbf{t}] f^{\mathrm{I}} \,\mathrm{d}s\right\} = 0. \tag{8b}$$

Notice that a discontinuity of stress is anticipated at the interfaces s_i.

If node I is an interior node, i.e. v_{I} lies wholly within v, then the last integral disappears from (8b). Then, because each nodal displacement is arbitrary, the discrete condition at the interior node follows:

$$-\int\int\int_{v_{\mathrm{I}}} \left[\frac{1}{\sqrt{g}} \frac{\partial}{\partial\theta^i}(s^{ij}\mathbf{G}_j) + \rho_0 \mathbf{f}\right] f^{\mathrm{I}} \,\mathrm{d}v + \int\int_{s_{(i)}} [{}_0n_i(s^{ij}_{+} - s^{ij}_{-})\mathbf{G}_j] f^{\mathrm{I}} \,\mathrm{d}s_{(i)} = 0. \tag{9}$$

If the stress is continuous, then the interface integrals vanish in (9) and the discrete condition asserts that a weighted average of the differential equation vanishes. The result is similar to a Galerkin averaging of the differential equation, *but* notice that the differential equation is averaged over v_{I} only. Moreover, according to (7a), if the derivatives are continuous, then a mean value of the differential equation vanishes. If the stress is discontinuous at the interfaces s_i, then equation (9) asserts that weighted averages of the jumps must be included with the average of the stress differential.

If the approximation of **V** is the simple tri-linear interpolation of (6), then the corresponding strain approximation in v_I is determined solely by the displacements of the nodes in the lattice of v_I. Therefore, if the body is elastic, the corresponding stress approximation is also determined solely by the nodal displacements of v_I. It follows that the equilibrium equation (9) contains no more than 81 displacement components, the displacement of the Ith node and its nearest neighbors.

If the stress is discontinuous on the interfaces, then a stress does not exist there in the usual sense. To establish a meaningful approximation for the stress at a node, we consider the exterior node J as depicted in Figs. 3 and 5. For simplicity, we suppose that the boundary lies in a coordinate surface; the portion s_J undergoes a prescribed virtual displacement proportional to $\delta \mathbf{V}_J$. The discrete condition of equilibrium follows from (8):

$$\mathbf{R}^J \equiv \iint_{s_J} \mathbf{t} f^J \, ds = \iint_{s_J} {}_0n_i s^{ij}\mathbf{G}_j f^J \, ds - \iiint \left[\frac{1}{\sqrt{g}}(\sqrt{g}s^{ij}\mathbf{G}_j)_{,i} + \rho_0 \mathbf{f} \right] f^J \, dv$$

$$+ \iint_{s_{(\alpha)}} [{}_0n_i(s^{ij}_{+} - s^{ij}_{-})\mathbf{G}_j] f^I \, ds_{(\alpha)}. \qquad \text{(10a,b)}$$

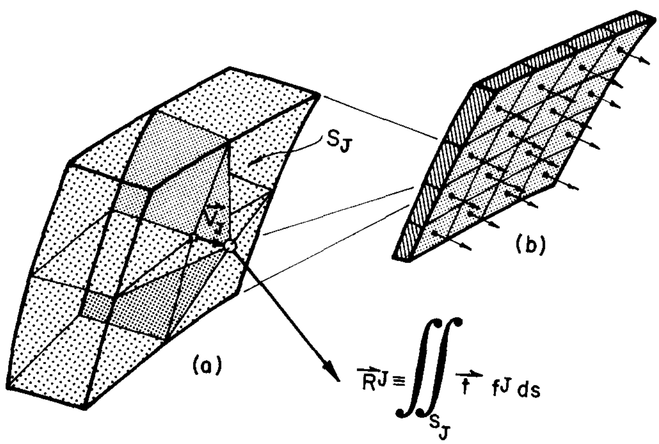

FIG. 5.

The index α of the final integral signifies that the integration applies only to the two *interfaces* through J. The generalized force $\mathbf{R}^J$ is defined by (10a). If the traction **t** on the boundary is continuous, then, according to (7b),

$$\frac{\mathbf{R}_J}{s_J} = \mathbf{t}_m$$

where $\mathbf{t}_m$ signifies a mean value.

The traction **t** may be approximated in the form:

$$\mathbf{t} = g_K(\theta^1, \theta^2, \theta^3)\mathbf{t}^K$$

where $\mathbf{t}^K$ denotes a nodal value, and summation over all nodes is implied. A necessary condition follows from the virtual work upon s_J:

$$\lim_{s_J \to 0} \frac{1}{S_J} \iint_{s_J} g_K f^J \, ds = \delta^J_K$$

where δ^1_K denotes the Kronecker delta. Appropriate forms of the function g_K have been investigated by Brauchli.†

If we split the region v_{I} along one interface and introduce the mean stress $\mathbf{t}_m$ upon each of the contiguous interfaces, then the condition (8) asserts the continuity of such mean stresses.

It is interesting to compare the conditions of the continuum theory with those of a discrete approximation: in the continuum theory, the virtual displacement is an arbitrary continuous function and, therefore, the work must vanish in an arbitrarily small neighborhood within v and, independently, on s. By contrast, the conditions of the approximation require that the virtual work vanish in each small, but finite, region v_{I} within v and, independently, in a shallow layer along the boundary s, as depicted in Fig. 5(b), and, moreover, in each piece v_J of that layer. The thickness of the layer diminishes as the mesh is refined and the approximation approaches the continuum.

Approximation by polynomials of higher-degree

A simple tri-linear polynomial is adequate for the approximation of some media. However, if the energy function depends upon higher derivatives of the displacement (in addition to the symmetrical strain-component γ_{ij}), then the simple interpolation is inadequate. An interpolating polynomial is inadequate if the required higher derivatives vanish identically. An adequate higher-order interpolation introduces nodal values of the derivatives, e.g. $(\mathbf{V}_{,i})_{\mathrm{I}}$, and the corresponding generalized forces. The latter are couples (and higher-order couples) akin to the couple-stresses of the continuum theory [25, 26].

Other stationary theorems and their discrete counterparts

Any stationary (or extremum) theorem of a continuum theory provides a basis for a discrete approximation. Theorems for finite deformation follow:

Washizu [27, 28] and Fraeijs de Veubeke [18] have presented a general stationary condition for *small* deformations. The theorem for finite deformations follows: the equilibrium conditions, strain–displacement relations and constitutive equations of a conservative body are the stationary conditions for the following functional W of the displacement vector $\mathbf{V}$, the stress tensor s^{ij} and the strain tensor γ_{ij}.

$$W \equiv \iiint_v \{U(\gamma_{ij}) + \omega_{\mathrm{B}}(V^i) - s^{ij}[\gamma_{ij} - \tfrac{1}{2}(\mathbf{g}_i \cdot \mathbf{V}_{,j} + \mathbf{g}_j \cdot \mathbf{V}_{,i} + \mathbf{V}_{,i} \cdot \mathbf{V}_{,j})]\} \, dv$$

$$+ \iint_{s_{\mathrm{I}}} \omega_{\mathrm{T}}(V^i) \, ds + \iint_{s_{\mathrm{II}}} \mathbf{t} \cdot (\mathbf{V} - \tilde{\mathbf{V}}) \, ds. \tag{11}$$

Here U is the free energy density (the temperature is assumed constant), ω_{B} is the potential of body forces, ω_{T} the potential of the tractions applied upon a portion s_{I} of the surface and $\tilde{\mathbf{V}}$ the displacement prescribed upon a portion s_{II}.

† His development of the complementary functions was conveyed to the author by Brauchli in private discussions.

To employ the general theorem, one approximates the displacement $\mathbf{V}$, the strain γ_{ij} and stress s^{ij} by spline functions. Such approximations must approach the required continuity and generality of their continuous counterparts as the mesh is refined. The resulting approximation of the integral W is rendered stationary with respect the nodal variations ($\delta \mathbf{V}_{\mathrm{I}}$, $\delta s_{\mathrm{I}}^{ij}$ and $\delta \gamma_{ij}^{\mathrm{I}}$). The resulting conditions are discrete counterparts of the continuum conditions, namely, weighted averages of the equilibrium equations, strain–displacement equations and constitutive equations.

The Hellinger–Reissner theorem [29, 30] has been used recently by Pister and Dunham [23] to formulate a discrete approximation. Instead of (11) we have the functional:

$$\overline{W} \equiv \iiint_{v} \{\overline{U}(s^{ij}) + \omega_{\mathrm{B}}(V^i) + \tfrac{1}{2}s^{ij}(\mathbf{g}_i \cdot \mathbf{V}_{,j} + \mathbf{g}_j \cdot \mathbf{V}_{,i} + \mathbf{V}_{,i} \cdot \mathbf{V}_{,j})\}\, \mathrm{d}v$$

$$+ \iint_{s_{\mathrm{I}}} \omega_{\mathrm{T}}(V^i)\, \mathrm{d}s + \iint_{s_{\mathrm{II}}} \mathbf{t} \cdot (\mathbf{V} - \tilde{\mathbf{V}})\, \mathrm{d}s. \tag{12}$$

Here $\overline{U}$ is the complementary energy density (again, the temperature is assumed constant). Corresponding to the variations of the displacement $\mathbf{V}$ and the stress s^{ij}, one obtains the conditions of equilibrium and the constitutive equations (stress–displacement relations). Again, spline approximations of the displacement and the stress tensor lead to the discrete conditions, weighted averages of the equilibrium equations and constitutive equations, respectively.

The Hellinger–Reissner theorem may also apply to certain plastic problems with appropriate modifications of the functional $\overline{W}$: if $\Delta\mathbf{V}$ denotes an incremental displacement, and if the plastic strain increment $\Delta\gamma_{ij}^{P}$ derives from a "plastic potential" $\overline{\overline{U}}$, i.e.

$$\Delta\gamma_{ij}^{P} = -\frac{\partial \overline{\overline{U}}}{\partial s^{ij}} \tag{13a, b}$$

then the theorem applies with the integral modified as follows:

$$\overline{W} \equiv \iiint_{v} \{\Delta\overline{U}(s^{ij}) + \overline{U}(s^{ij}) + \omega_{\mathrm{B}}(V^i) + \tfrac{1}{2}s^{ij}(\mathbf{G}_i \cdot \Delta\mathbf{V}_{,j} + \mathbf{G}_j \cdot \Delta\mathbf{V}_{,i})\}\, \mathrm{d}v$$

$$+ \iint_{s_{\mathrm{I}}} \omega_{\mathrm{T}}(V^i)\, \mathrm{d}s + \iint_{s_{\mathrm{II}}} \mathbf{t} \cdot (\mathbf{V} - \tilde{\mathbf{V}})\, \mathrm{d}s. \tag{14}$$

In practical applications, in the continuum theory or the discrete approximation, it may be necessary to cope with yield criteria and conditions for plastic loading. The latter often have a variational basis (e.g. an energy principle) but a yield criterion may not. However, the yield criterion can be imposed via a Lagrangian multiplier, as an auxiliary condition upon the stress field. Then the discrete counterpart is again a weighted average in the small finite region v_{I}.

The stationary theorem of Reissner [30] has been employed by Greene *et al.* [31] in order to relax the continuity requirements, which otherwise beset the analyses of Kirchhoff plates and shells.

PARTIAL APPROXIMATION

The shell as a partial finite-element

Many current problems in structural mechanics are concerned with shells. Approximations of shell theories can be achieved by the methods previously described. Furthermore, continuum theories of shells have much in common with such approximations. Indeed, one could say that most approximations of a shell are partial finite elements obtained by introducing polynomial approximations in one coordinate only, specifically, the thickness coordinate.

Suppose that the principle of virtual work is employed in the manner of Ritz, together with the following approximation:

$$\mathbf{V} \doteq {}_0\mathbf{V}(\theta^1, \theta^2) + {}_1\mathbf{V}(\theta^1, \theta^2)\theta^3. \tag{15}$$

Here θ^3 denotes the distance along the undeformed normal to a reference surface $\theta^3 = 0$. The upper and lower surfaces of the shell are defined by $\theta^3 = \pm h_\pm$. In effect, the approximation (15) is an interpolation in θ^3; the displacement $\mathbf{V}(\theta^1, \theta^2, \pm h_\pm)$ could replace ${}_N\mathbf{V}(\theta^1, \theta^2)$ $(N = 0, 1)$ in (15).

Here we do not restrict ourselves to a conservative body, but refer to the deformed state. Upper and lower surfaces belong to the deformed surface S_I on which tractions are prescribed, as does a portion of the edge defined by a curve C_I on $\theta^3 = 0$. For simplicity we denote length along C_I by θ'^2 and length along a normal curve by θ'^1. A prime (′) signifies a quantity associated with the local coordinates (θ'^i) at the boundary.

Let us define stress resultants:

$${}_N\mathbf{m}^\alpha \equiv \int_{-h_-}^{h_+} \tau^{\alpha j}\mathbf{G}_j \sqrt{\left(\frac{G}{A}\right)}(\theta^3)^N \,\mathrm{d}\theta^3 \tag{16a}$$

$${}_N\boldsymbol{\tau} \equiv (N+1)\int_{-h_-}^{h_+} \tau^{3j}\mathbf{G}_j \sqrt{\left(\frac{G}{A}\right)}(\theta^3)^N \,\mathrm{d}\theta^3 \tag{16b}$$

where $N = 0, 1$, but could have an extended range. Similarly, we define edge resultants:

$${}_N\mathbf{m}'^1 \equiv \int_{-h_-}^{h_+} \tau'^{\alpha j}\mathbf{G}'_j \sqrt{\left(\frac{G}{A}\right)}(\theta^3)^N \,\mathrm{d}\theta^3. \tag{17}$$

Next, we define resultants of surface tractions, ${}_N\mathbf{c}$, and body forces, ${}_N\mathbf{b}$:

$${}_N\mathbf{c} \equiv \tau_+^{3j}\mathbf{G}_{j+}\sqrt{\left(\frac{G_+}{A}\right)}h_+^N - (-1)^N\tau_-^{3j}\mathbf{G}_{j-}\sqrt{\left(\frac{G_-}{A}\right)}h_-^N \tag{18a}$$

$${}_N\mathbf{b} \equiv \int_{-h_-}^{h_+} \rho\mathbf{f}\sqrt{\left(\frac{G}{A}\right)}(\theta^3)^N \,\mathrm{d}\theta^3. \tag{18b}$$

The resultant of external forces is

$${}_N\mathbf{f} \equiv {}_N\mathbf{b} + {}_N\mathbf{c}. \tag{19}$$

The shell equivalents of (4a) and (4b) are obtained by introducing (15) into (4a). Upon integrating with respect to θ^3 and employing the definitions (16)–(19), we obtain

$$\delta W = \int\int_S [{}_N\mathbf{m}^\alpha \cdot \delta_N \mathbf{V}_{,\alpha} + {}_0\boldsymbol{\tau} \cdot \delta_1 \mathbf{V} - {}_N\mathbf{f} \cdot \delta_N \mathbf{V}]\, dS - \int_{C_1} {}_N\mathbf{m}'^1 \cdot \delta_N \mathbf{V}\, d\theta'^2 \tag{20a}$$

$$\delta W = -\int\int_S \{[(\sqrt{A}\,{}_0\mathbf{m}^\alpha)_{,\alpha} + {}_0\mathbf{f}] \cdot \delta_0 \mathbf{V} + [(\sqrt{A}\,{}_1\mathbf{m}^\alpha)_{,\alpha} + {}_1\mathbf{f} - {}_0\boldsymbol{\tau}] \cdot \delta_1 \mathbf{V}\}\, d\theta^1\, d\theta^2$$

$$+ \int_{C_1} \{[(\mathbf{A}_\alpha \cdot \hat{\lambda}_1)_0 \mathbf{m}^\alpha - {}_0\mathbf{m}'^1] \cdot \delta_0 \mathbf{V} + [(\mathbf{A}_\alpha \cdot \hat{\lambda}_1)_1 \mathbf{m}^\alpha - {}_1\mathbf{m}'^1] \cdot \delta_1 \mathbf{V}\}\, d\theta'^2 \tag{20b}$$

Here S signifies the deformed reference surface $\theta^3 = 0$.

The equilibrium conditions implied by (20b) are *precisely* (a) the three-dimensional condition integrated through the thickness and (b) the three-dimensional condition multiplied by θ^3 and integrated through the thickness. If the approximation (15) contains higher powers, then the additional equilibrium equations are higher moments of the three-dimensional conditions. In short, the equilibrium equations of the resulting shell theory are weighted averages through the finite thickness, much as a three-dimensional approximation produces a weighted average through a finite-element.

The Kirchhoff–Love hypothesis asserts that

$$\delta\gamma_{3\alpha}]_S \equiv \tfrac{1}{2}(\mathbf{A}_3 \cdot \delta_0 \mathbf{V}_{,\alpha} + \mathbf{A}_\alpha \cdot \delta_1 \mathbf{V}) = 0 \tag{21a, b}$$

where

$$\mathbf{A}_i = \mathbf{G}_i(\theta^1, \theta^2, 0).$$

Equation (21b) serves to express tangential components of $\delta_1\mathbf{V}$ in terms of the normal components of $\delta_0\mathbf{V}_{,\alpha}$. In the continuum theory this constraint reduces the number of differential equations of equilibrium. In the discrete approximation of such continuum theory the normal component of $\delta_0\mathbf{V}_{,\alpha}$ must be continuous, i.e. the surface S must be smooth. It is simpler to employ the unconstrained continuum theory and then, to impose (21b) only at points of the interelement boundaries [32, 33].

Approximating the shell by spline function

The vectors ${}_N\mathbf{V}$ can be approximated in the manner of (5):

$${}_N\mathbf{V}(\theta^1, \theta^2) = f^I(\theta^1, \theta^2)_N \mathbf{V}_I. \tag{22}$$

Then the Ritz procedure produces conditions of equilibrium which, like (9) and (10), are weighted averages of the continuum equations, integrals extending over the four elements contiguous at the Ith node. Also, if the approximations of the stress resultants are discontinuous, then jumps are added to the derivatives in the integrands.

In the unconstrained theory of (15), ${}_0\mathbf{V}$ and ${}_1\mathbf{V}$ are independent, and Lagrangian (bilinear) interpolation achieves continuity of the displacement. Discrete constraints can be imposed upon the transverse shear strain at points of the interfaces. Such constraints insure that the approximation converges to the Kirchhoff–Love theory. If these are enforced via Lagrangian multipliers then the multipliers are generalized forces of constraint.

Other stationary principles may serve as the basis of a shell theory and then a discrete approximation, i.e. a finite-element formulation. For example, the general statement of stationary potential energy can be employed via the energy integral (11). Firstly, one must introduce the basic approximations of the shell theory: the approximation (15) must be augmented by suitable approximations of the stress and strain tensors; approximations were introduced by Reissner [34] and Naghdi [35] in the linear theory. Similar approximations are applicable to problems of large deflections with small strain. The results are the equilibrium equations, strain–displacement relations and constitutive equations of the shell theory.

APPROXIMATING THE LOADING PATH

Incremental displacements, corrections and unstable states

A discrete model of the conservative structural system is governed by a system of non-linear algebraic equations:†

$$N^Q(V_Q;\lambda) = 0. \tag{23}$$

For a stable state $(\bar{V}_Q;\bar{\lambda})$, an incremental displacement is approximated by the linear system:

$$\overline{\frac{\partial N^Q}{\partial V_R}}\Delta V_R = -\overline{\frac{\partial N}{\partial \lambda}}\Delta\lambda. \tag{24}$$

Successive increments generate the equilibrium path $V_Q(\lambda)$. However, the approximate path strays from the solution of (23). Various schemes can be employed to improve the approximation: for example, better estimates for the mean values of the derivatives [11, 12] and the introduction of higher derivatives [36, 37]. Here we suggest a simple approach which can be effective [38]:

Let the barred ($\bar{\ }$) coefficients of (24) be the values of the reference state $(\bar{V}_Q;\bar{\lambda})$ (e.g. point M in Fig. 6). Then the solution of the system (24) provides an approximation $(\bar{V}_Q+\Delta V_Q;\bar{\lambda}+\Delta\lambda)$ of the nearby equilibrium state (e.g. point N in Fig. 6). Substituting these values into the left side of (23) determines the errors R^Q

$$N^Q(\bar{V}_Q+\Delta V_Q;\bar{\lambda}+\Delta\lambda) \equiv R^Q. \tag{25}$$

Corrections $\Delta\tilde{V}_Q$ may be obtained by the Newton–Raphson method, i.e.

$$\overline{\overline{\frac{\partial N^Q}{\partial V_R}}}\Delta\tilde{V}_R = -R^Q. \tag{26}$$

Here the double bar ($\bar{\bar{\ }}$) signifies evaluation at the current state (e.g. point N of Fig. 6). The correction procedure of (26) may be repeated until the errors R^Q are made as small as desired. Observe that the coefficients in the left sides of (24) and (26) are the same variables, but evaluated at different states, i.e. the matrices for the incremental and corrective procedures are computed in the same way.

Figure 6 depicts the increment $(\Delta V;\Delta\lambda)$ which extrapolates the path from M to N, followed by the correction $(\Delta\tilde{V};0)$. Since the stiffness (slope) can change drastically during loading, uniform loading steps produce varying incremental displacements. Moreover, at a bifurcation point P or limit point Q, there exists a solution for $\Delta\lambda = 0$. Therefore, it is

† To simplify notations, nodal components $V_i^I (I = 1, \ldots, m)$ are relabeled $V_Q(Q = 1, \ldots, 3m = n)$.

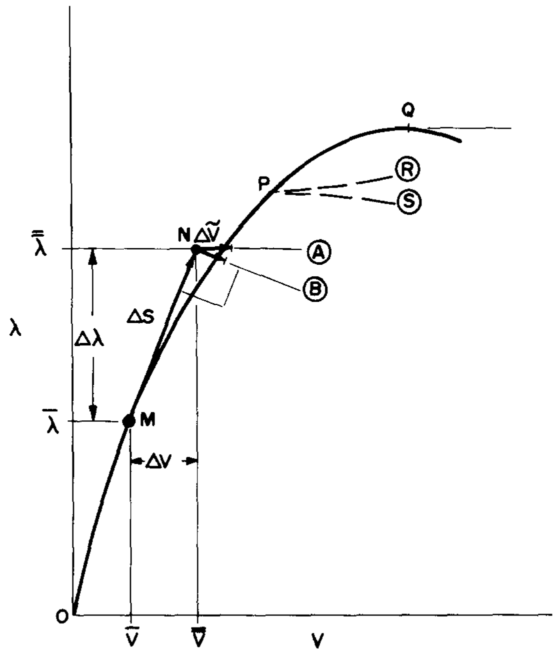

FIG. 6.

desirable, even necessary, to prescribe an incremental parameter other than the load. To this end, we define a generalized arc-length S:

$$\mathrm{d}S^2 \equiv \mathrm{d}V_Q\,\mathrm{d}V_Q + \mathrm{d}\lambda\,\mathrm{d}\lambda.$$

Now, suppose that a prefix M enumerates a particular step. Then according to (24)

$$_{\mathbf{M}}\left[\frac{\partial N^Q}{\partial V_R}\right]\Delta_{\mathbf{M}}V_R + {}_{\mathbf{M}}\left[\frac{\partial N^Q}{\partial V_R}\right]\Delta_{\mathbf{M}}\lambda = 0. \tag{27a}$$

The system (27a) is now augmented by the equation:

$$[\Delta_{\mathbf{M}-1}V]\Delta_{\mathbf{M}}V + [\Delta_{\mathbf{M}-1}\lambda]\Delta_{\mathbf{M}}\lambda = \Delta S^2. \tag{27b}$$

If $Q = 1, \ldots, n$, then (27a, b) is a linear system of $(n+1)$ unknowns $(\Delta V_Q, \Delta\lambda)$ while ΔS is the prescribed loading parameter. Instead of (26) we determine a correction "orthogonal" to the increment $(\Delta V_Q, \Delta\lambda)$:

$$_{\mathbf{M}+1}\left[\frac{\partial N^Q}{\partial V_R}\right]\Delta_{\mathbf{M}}\tilde{V}_R + {}_{\mathbf{M}+1}\left[\frac{\partial N^Q}{\partial \lambda}\right]\Delta_{\mathbf{M}}\tilde{\lambda} = -{}_{\mathbf{M}+1}[N^Q] \tag{28a}$$

$$[\Delta_{\mathbf{M}}V_R]\Delta_{\mathbf{M}}\tilde{V}_R + [\Delta_{\mathbf{M}}\lambda]\Delta_{\mathbf{M}}\tilde{\lambda} = 0. \tag{28b}$$

The correction of (28) is illustrated at B of Fig. 6, while the correction of (26) is depicted at A.

Equation (27b) can hold only if the path is smooth. At the bifurcation point P,

$$\left|\frac{\partial N^Q}{\partial V_R}\right| = 0.$$

Here the ensuing step is determined by the eigenvector of the homogeneous system:

$$\frac{\overline{\partial N^Q}}{\partial V_R} \Delta V_R = 0.$$

Having turned the corner at P, we proceed as before. However, in most cases we require only the computations to the critical load of P and, perhaps, an answer to the question of stability at the critical load. The latter can be answered by the solution of additional linear systems as given by Koiter [24].

Examples

Numerous authors have utilized incremental loading to obtain approximations for nonlinear problems: Argyris [3, 7, 38] has combined incremental loads and finite elements to treat large deflections and plastic deformations. Martin [39] employed increments in his work on large deflections and instability. Hofmeister *et al.* [40], Zudans [41] and Marcal [42] have used the incremental approach in plasticity, and Oden [43] has employed incremental loads in nonlinear elasticity. More references are cited in the bibliographies of Argyris [7] and Oden [8].

Incremental loading has been applied to finite elastic deformation with limited success. Boedecker [44], Oden and Key [45] have encountered computational difficulties at large strains. Their approximations grow erratic in a manner described by Richtmeyer and Morton [46] and illustrated by the approximation of waves, wherein time is an independent variable, much as the load in our problems.

The finite deflections, buckling and postbuckling of a circular arch provide an example of the procedures described herein. Details of such computations were presented previously [47]. The deflection of a steep arch follows a path like OP of Fig. 6; buckling occurs by sidesway along a path like PR. A very shallow arch deflects symmetrically and snaps-through at a limit point like Q.

Another problem which illustrates the use of our loading parameter is that of a toroidal membrane composed of Mooney material [48] and subjected to internal pressure p. The toroid and pertinent quantities are shown in Fig. 7. The extension ratios in the longitudinal and meridional directions are

$$\lambda_1 = \frac{Q}{\rho - c \sin \beta}, \qquad \lambda_2 = \frac{d\xi}{d\eta}.$$

The simplest approximation is a two-degree-of-freedom model: the deformed cross-section is assumed circular with radius C and center at distance R from the axis. Particles are assumed to maintain their meridional angle. Then average values of λ_1 and λ_2 are

$$\bar{\lambda}_1 = \frac{R}{\rho}, \qquad \bar{\lambda}_2 = \frac{C}{c}.$$

Results were obtained with a membrane thickness $h = 0{\cdot}001c$ and the Mooney constants $C_2 = 0{\cdot}1C_1$. Plots of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ vs. pressure p are shown in Fig. 8. These results agree with those displayed by Kydoniefs and Spencer [49].

The symmetrical deflection of a shallow shell provides another illustration: the truncated cone of Fig. 9 is deflected by an axial load distributed uniformly on free edges. Differential equations and a numerical solution were obtained earlier [50]. Now, if those equations are

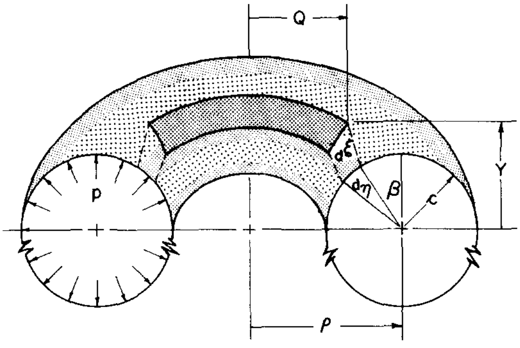

FIG. 7.

approximated by simple differences and the incremental procedure of Fig. 6(B) is used, then the load–deflection curves of Fig. 9 are obtained. The upper curve is obtained with 100 steps and strays from the better curve obtained with 1000 steps. The latter curve is less precise than the earlier result [50], because only five sub-intervals of the meridian were used in the present computations.

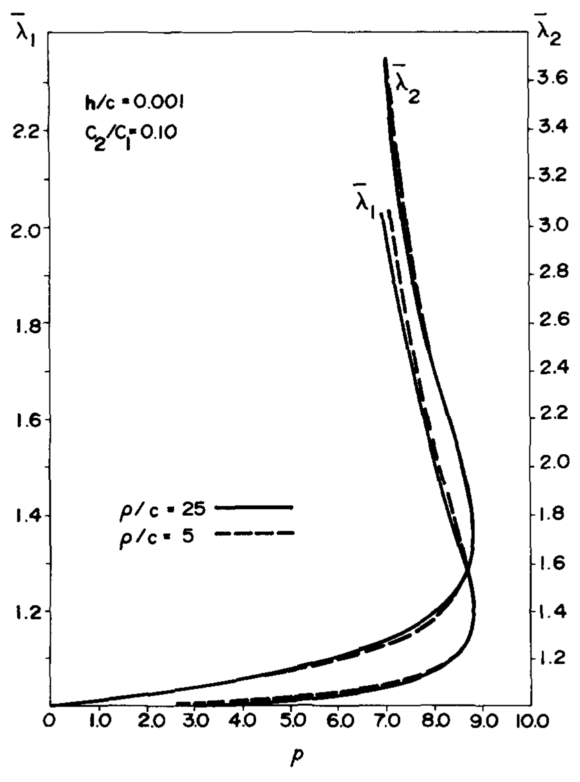

FIG. 8.

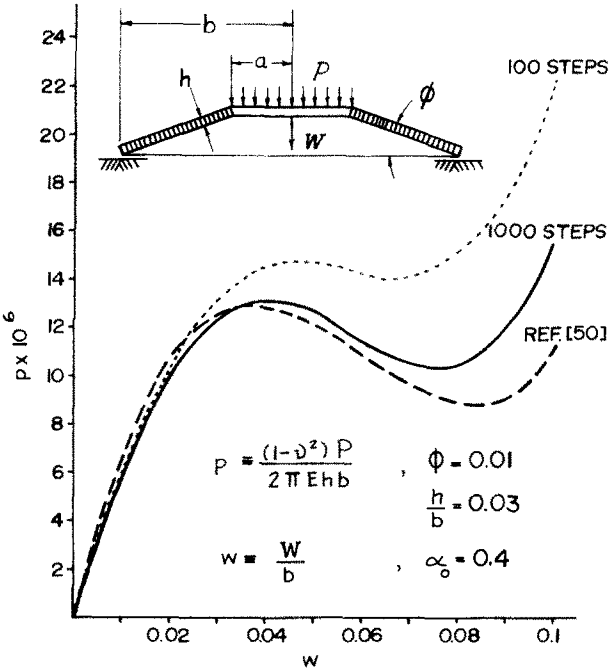

FIG. 9.

CONCLUSIONS

The familiar notions of interpolations and extrapolation provide effective bases for numerical studies of many nonlinear problems of solid mechanics.

The use of the variational theorems in the manner of Rayleigh–Ritz serves to relate the Euler equations and their algebraic counterparts.

The introduction of a generalized arc-length in a configuration-load space provides an effective loading parameter which facilitates the approximation of nonlinear solutions.

The geometrical interpretations provide some insight to the structure of piece-wise approximations.

SUMMARY

The notions of interpolation and extrapolation provide access to the nonlinear problems of elastic and inelastic bodies. The former is the basis for spline approximations which define the configuration in terms of discrete nodal values. The latter provides the basis for approximating the nonlinear path.

Algebraic equations governing the approximation are derived by stationary theorems for finite deformations. The discrete analogues of the continuum equations are interpreted as weighted averages of their differential counterparts. A general stationary theorem produces the complete system of discrete analogues: the equilibrium, kinematical and constitutive equations.

The method of incremental-loading is modified by introducing a generalized arc-length in a configuration-load space. The modified procedure is convenient for computations at

limit points and at bifurcation points. The method is illustrated by selected examples of large deformations and instabilities.

Essential features of the presentation are (1) the relation of the Euler equations to their algebraic counterparts, (2) a review of the stationary theorems for finite deformations, and (3) the introduction of a generalized arc-length to facilitate the procedure of incremental loading.

Details of the developments are omitted to avoid a profusion of symbols which so often obscure the salient features. Selected references merely provide the basis of previous and/or related developments; many other contributions are cited in the extensive bibliographies of Felippa and Clough [6], Argyris [7] and Oden [8].

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APPENDIX

Finite element viewpoint

In the method of finite elements it is customary to adopt the following viewpoint: each element is isolated. The principle of virtual work (or Lagrange's equations) serves to express the generalized external forces in terms of the nodal (corner) displacements. Subsequently, Newton's law of reaction is imposed at contiguous corners. It follows that the interactions between elements do no net work. Consequently, the latter viewpoint produces the same results as our viewpoint. However, it is also interesting to consider the meanings of the approximate conditions imposed upon the element: suppose that the quadrilateral element of Fig. 10 is subject to a virtual displacement in accordance with (6), specifically,

$$\delta\mathbf{V} = \delta\mathbf{A} + \delta\mathbf{B}_i\theta^i + \tfrac{1}{2}\delta\mathbf{C}_{ij}\theta^i\theta^j + \delta\mathbf{D}\theta^1\theta^2\theta^3$$

where θ^i signifies a local coordinate originating at the particle I of the element. No strain is

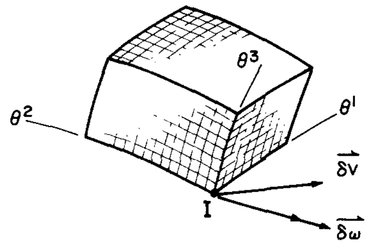

FIG. 10.

caused by the constant $\delta\mathbf{A}$, the displacement of particle I. A condition of equilibrium, corresponding to the virtual displacement $\delta\mathbf{A}$, requires that the resultant of all external forces vanishes. The constants $\delta\mathbf{B}_i$ can be replaced by the components:

$$\delta\gamma_{ij}]_{\mathrm{I}} \equiv \tfrac{1}{2}(\mathbf{G}_i]_{\mathrm{I}}\,.\,\delta\mathbf{B}_j+\mathbf{G}_j]_{\mathrm{I}}\,.\,\delta\mathbf{B}_i)$$

$$\delta\omega_{ij}]_{\mathrm{I}} \equiv \tfrac{1}{2}(\mathbf{G}_i]_{\mathrm{I}}\,.\,\delta\mathbf{B}_j - \mathbf{G}_j]_{\mathrm{I}}\,.\,\delta\mathbf{B}_i).$$

The latter is associated with a rigid-body rotation at particle I:

$$\delta\boldsymbol{\omega} \equiv \tfrac{1}{2}\varepsilon^{jip}\omega_{ij}\mathbf{G}_p]_{\mathrm{I}}$$

A condition associated with the virtual displacement ($\delta\mathbf{A}$) requires the vanishing of external *force*; a condition associated with the virtual rotation ($\delta\boldsymbol{\omega} = \frac{1}{2}\varepsilon^{jip}\omega_{ij}\mathbf{G}_p$) *approaches* the condition of vanishing *moment* and the symmetry of the stress tensor; the remaining conditions associated with the linear terms ($\delta\gamma_{ij}$) of our approximation approach the constitutive equations of the continuum theory. The higher-degree terms of the approximation are needed to achieve continuity at the interface.

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Абстракт—Используются интерполяция и экстраполяция для апироксимации полей в нелинейной теории твердых тел. Применяются узловые точки в пространстве положения и нагрузки. Сплошные поля заменяются узловыми величинами. Интерполяция между узлами/экстраполяция касающаяся ки/ определяет сплошную аппроксинацию. Дифференциальные уравнения сплошной среды заменяются алгебраическими уровнениями дискретной системы. Нелинейиые уравнения заменяются последовательностью линейных уравнений, так как нелинейные траектория приближается линейными сегментами.

Используются вариационные теоремы, как основы алгебраических формулировок, которые обладают дмскретным приближением. Алгебраические уравнения отнесённые к их дифференциальным соответствующим частям.

Принимается обобщенная длина арки в пространстве конфигурация—нагрузка, с целью улучшения расчетов приращений, вблизи граничных точек. Используется длина арки в смысле параметра нагрузки, В некоторых иллюстративных примерах.

В приложении даётся точка зрения конечных элементов и условия непрерывности, которые обеспечивают эквивалентность этих методов.